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GENERAL EXPRESSIONS OF THE DEVELOPMENT IN SERIES OF THE COORDINATES OF A CELESTIAL BODY

D. Cipolletti

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Cipolletti, D.

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GENERAL EXPRESSIONS OF THE DEVELOPMENT IN SERIES OF THE COORDINATES OF A CELESTIAL BODY

Calling the function of the solar accelerating force, F; the time multiplied by $\sqrt{g}$ as ascertained on page 7, line 9, of the very fine posthumons memorandum by Ottaviano Fabrizio Mossotti "On the determination of the orbits of the celestial bodies" (1) t; meaning by g, that constant depending on the attractive force of the sun which is defined on page 1, lines 24 - 25, page 2, lines 1 - 8, of the "Theory of the movement of the celestial bodies" by Charles Frederick Gauss; the three components of said accelerating force, at any one instant in which r be the radius vector and x, y, z, the coordinates of a celestial body are:

$$(1) \qquad \frac{d^2x}{dt^2} = -F.\frac{x}{r}, \quad \frac{d^2y}{dt^2} = -F.\frac{y}{r}, \quad \frac{d^2z}{dt^2} = -F.\frac{z}{r}$$

Or, calling u the measure of F in respect of r:

$$(2) \qquad \frac{d^2x}{dt^2} = -u.x, \quad \frac{d^2y}{dt^2} = -u.y, \quad \frac{d^2z}{dt^2} = -u.z$$

In which, u is the function of r; and r, x, y, z, are functions of the time.

For the sake of simplicity we shall represent the three equations, (2), only by the third: then, if x_0, y_0, z_0,

(1) Science is indebted to the most Christian Director of this Observatory, for knowledge of such a memorandum by the late eminent Professor of Pisa. Thanks to it, orbits of the celestial bodies in the most usual case of the data in coordinates and in time of three observations being known, are calculated by work that proves to be of the utmost facility as compared with that which would have to be done, if those orbits were to be calculated by other methods or with other systems of equations.

are the coordinates of the celestial body considered in the instant of an observation; the coordinates of the same body in an instant next to the first instant are: 0 being the interval

$$(3)\ Z = z_0 + \left(\frac{dz}{dt}\right)_0 \theta + \left(\frac{d^2z}{dt^2}\right)_0 \cdot \frac{\theta^2}{1.2} + \left(\frac{d^3z}{dt^3}\right)_0 \cdot \frac{\theta^3}{1.2.3} + \dots$$

where, the $\left(\frac{dz}{dt}\right)_0$ and the $\left(\frac{d^2z}{dt^2}\right)_0$ are the velocities, and the accelerating forces according with the three axes in that same first instant.

It being now the case that

$$(4)\quad \left\{ \begin{aligned} \frac{d^2z}{dt^2} &= - u \cdot z \\ \frac{d^3z}{dt^3} &= - \frac{d \cdot uz}{dt} \\ \frac{d^4z}{dt^4} &= - \frac{d^2 \cdot uz}{dt^2} \\ \dots & \dots \end{aligned} \right.$$

The (3) assume the form

$$(5)\quad Z = z_0 + \left(\frac{dz}{dt}\right)_0 \theta - (uz)_0 \frac{\theta^2}{1.2} - \left(\frac{d \cdot uz}{dt}\right)_0 \cdot \frac{\theta^3}{1.2.3} - \text{etc.}$$

which I propose to reduce to the binomial

$$(6)\quad Z = A_z z_0 + B_z \cdot \left(\frac{dz}{dt}\right)_0$$

fixing the law of formation of the expressions A_z, and B_z, which must clearly be ordinated by the powers of the interval

In the equations (5) we have:

$$(7)\quad \left\{ \begin{aligned} \frac{d \cdot uz}{dt} &= z \frac{du}{dt} + u \cdot \frac{dz}{dt} \\ \frac{d^2 \cdot uz}{dt^2} &= z \frac{d^2u}{dt^2} + 2 \frac{du}{dt} \cdot \frac{dz}{dt} + u \frac{d^2z}{dt^2} \end{aligned} \right.$$

etc., etc.

Which, because of the (4) become:

$$
(8)\quad \begin{cases}
\dfrac{d\,.\,uz}{dt} = z\,.\,\dfrac{du}{dt} + u\,.\,\dfrac{dz}{dt} \\[2mm]
\dfrac{d^2.\,uz}{dt^2} = z\,.\,\dfrac{d^2u}{dt^2} + 2\,\dfrac{du}{dt}\,.\,\dfrac{dz}{dt} - u^2 z \\[2mm]
\dfrac{d^3.\,uz}{dt^3} = z\,.\,\dfrac{d^3u}{dt^3} + 3\,\dfrac{d^2u}{dt^2}\,\dfrac{dz}{dt} - 3uz\,.\,\dfrac{du}{dz} - u\left(z\,\dfrac{du}{dt} + u\,\dfrac{dz}{dt}\right)
\end{cases}
$$

etc., etc.

And, ordering them in the binomial form

$$
Az + B\,\frac{dz}{dt}
$$

we shall obtain

$$
(9)\quad \begin{cases}
\dfrac{d\,.\,uz}{dt} = \dfrac{du}{dt}\,.\,z + u\,\dfrac{dz}{dt} \\[2mm]
\dfrac{d^2\,.\,uz}{dt^2} = \left(\dfrac{d^2u}{dt^2} - u^2\right) z + 2\,\dfrac{du}{dt}\,.\,\dfrac{dz}{dt} \\[2mm]
\dfrac{d^3\,.\,uz}{dt^3} = \left(\dfrac{d^3u}{dt^3} - 4u\,\dfrac{du}{dt}\right) z + \left(3\,\dfrac{d^2u}{dt^2} - u^2\right)\dfrac{dz}{dt} \\[2mm]
\dfrac{d^4\,.\,uz}{dt^4} = \left(\dfrac{d^4u}{dt^4} - 7u\,\dfrac{d^2u}{dt^2} - 4\left(\dfrac{du}{dt}\right)^2 + u^3\right) z + \left(4\,\dfrac{d^3u}{dt^3} - 6u\,\dfrac{du}{dt}\right)\dfrac{dz}{dt} \\[2mm]
\dfrac{d^5\,.\,uz}{dt^5} = \left(\dfrac{d^5u}{dt^5} - 11\,.\,u\,\dfrac{d^3u}{dt^3} - 15\,\dfrac{du}{dt}\,.\,\dfrac{d^2u}{dt^2} + 9u^2\,\dfrac{du}{dt}\right) z + \left(5\,\dfrac{d^4u}{dt^4} - 13u\,\dfrac{d^2u}{dt^2} - 10\left(\dfrac{du}{dt}\right)^2 + u^3\right)\dfrac{dz}{dt} \\[2mm]
\dfrac{d^6.\,uz}{dt^6} = \left(\dfrac{d^6u}{dt^6} - 16\,\dfrac{ud^6u}{dt^4} - 26\,\dfrac{du}{dt}\,\dfrac{d^3u}{dt^3} + 28u\left(\dfrac{du}{dt}\right)^2 + 22u^2\,\dfrac{d^2u}{dt^2} - 15\left(\dfrac{d^2u}{dt^2}\right)^2 - u^4\right) z + \left(6\,\dfrac{d^5u}{dt^5} - 24u\,\dfrac{d^3u}{dt^3} - 38\,\dfrac{du}{dt}\,.\,\dfrac{d^2u}{dt^2} + 12u^3\,\dfrac{du}{dt}\right)\dfrac{dz}{dt}
\end{cases}
$$

etc.

And the law by which these derivatives proceed is the following: "the coefficient of z of each equation, equals the analogous derivative of the coefficient of center of the antecedent equation, less the coefficient of $\left.\frac{dz}{dt}\right|$ of the same antecedent equation multiplied by u; and the coefficient of $\left.\frac{dz}{dt}\right|$ equals the coefficient of z of the antecedent equation, plus the analogous derivative of the coefficient of center of the same antecedent equation."

But this law of derivation is not very suitable; and it is best to convert it into the one that follows:

It being the case that

$$\left.\begin{array}{l} a_1 = \frac{du}{dt} \\ b_1 = u \end{array}\right| \quad \text{etc., etc.}$$

The equations (9) are put in the form

$$(10) \quad \left\{\begin{array}{l} \frac{d \,.\, uz}{dt} = a_1 \, z + b_1 \,.\, \frac{dz}{dt} \\ \frac{d^2 \,.\, uz}{dt^2} = a_2 \, z + b_2 \,.\, \frac{dz}{dt} \\ \frac{d^3 \,.\, uz}{dt^3} = a_3 \, z + b_3 \,.\, \frac{dz}{dt} \\ \frac{d^4 \,.\, uz}{dt^4} = a_4 \, z + b_4 \,.\, \frac{dz}{dt} \end{array}\right|$$

and indicating the "Derivative" by D, we shall have by virtue of the law expounded above on formation of the coefficients a, and b,

$$
(11)\quad \left\{\begin{aligned}
a_2 &= D \,.\, a_1 - u \,.\, b_1 \\
a_3 &= D \,.\, a_2 - u \,.\, b_2 \\
a_4 &= D \,.\, a_3 - u \,.\, b_3 \\
a_5 &= D \,.\, a_4 - u \,.\, b_4 \\
&\ldots\ldots\ldots\ldots \\
b_2 &= a_1 + D \,.\, b_1 \\
b_3 &= a_2 + D \,.\, b_2 \\
b_4 &= a_3 + D \,.\, b_3 \\
b_5 &= a_4 + D \,.\, b_4
\end{aligned}\right.
$$

from which is obtained:

$$
(12)\quad \left\{\begin{aligned}
a_2 &= D \,.\, a_1 - u \,.\, b_1 \\
a_3 &= D \,.\, a_2 - u\,(a_1 + D \,.\, b_1) \\
a_4 &= D \,.\, a_3 - u\,(a_2 + D \,.\, a_1 + D^2 \,.\, b_1) \\
a_5 &= D \,.\, a_4 - u\,(a_3 + D \,.\, a_2 + D^2 \,.\, a_1 + D^3 \,.\, b_1) \\
a_6 &= D \,.\, a_5 - u\,(a_4 + D \,.\, a_3 + D^2 \,.\, a_2 + D^3 \,.\, a_1 + D^4 \,.\, b_1)
\end{aligned}\right.
$$

etc. etc.

$$
(13)\quad \left\{\begin{aligned}
b_2 &= D \,.\, b_1 + a_1 \\
b_3 &= D \,.\, b_2 + D \,.\, a_1 - u b_1 \\
b_4 &= D \,.\, b_3 + D \,.\, a_2 - u\,(a_1 + D \,.\, b_1) \\
b_5 &= D \,.\, b_4 + D \,.\, a_3 - u\,(a_2 + D\, a_1 + D^2 \,.\, b_1) \\
b_6 &= D \,.\, b_5 + D \,.\, a_4 - u\,(a_3 + D\, a_2 + D^2\, a_1 + D^3 \,.\, b_1)
\end{aligned}\right.
$$

etc., etc.,

or generally

$$(14)\begin{cases} a_n = D.a_{n-1} - u\left(a_{n-2} + D.a_{n-3} + D^2.a_{n-4} + \ldots + D^{n-2}.b_1\right) \\ b_n = D.b_{n-1} + D.a_{n-2} - u\left(a_{n-3} + D.a_{n-4} + D^2.a_{n-5} + \ldots + D^{n-3}.b_1\right) \end{cases}$$

And these equations which are easily predictable, completely solve the problem under consideration whatever the fraction of the solar attraction be. And in fact substituting in (5) in place of the derivatives

$$\frac{d.uz}{dt},\ \frac{d^2.uz}{dt^2},\qquad \text{etc.}$$

their values as given by (10), these transform themselves into

$$(15)\begin{cases} Z = \left[1 - u\,\frac{\theta^2}{1.2} - \theta^2\left(\frac{1}{2.3}\,a_1\,.\,\theta + \frac{1}{3.4}\,a_2\,.\,\frac{\theta^2}{1.2} + \text{ecc.},\right)\right]z_0 \\ +\left[\quad 1 - \theta^2\left(\frac{1}{2.3}\,b_1\,.\,\theta + \frac{1}{3.4}\,b_2\,.\,\frac{\theta^2}{1.2} + \text{ecc.},\right)\right]\left(\frac{dz}{dt}\right)_0 \end{cases}$$

which (here/recall -- and this has been established above -- that the Z represents the development of all the three coordinates X, Y, and Z) have the law of any coefficient a and b whatsoever, completely determined by the (14), and represent the integrals of the (2) in function of the interval θ, and of the six constants

$$x_0\ ,\ y_0\ ,\ z_0$$
$$\left(\frac{dx}{dt}\right)_0,\ \left(\frac{dy}{dt}\right)_0,\ \left(\frac{dz}{dt}\right)_0$$

which can be considered to be the six constants that will assertain to the integration of the same equations (2), derivatives of any order of 2°.

Royal Observatory, at Florence,
15 July 1872

N.B. The continuation of the paper "Applications of the Principle of Newton" etc., will be published in the next fascicule.